Abstract Definitions of all the Substitution Groups whose Degrees do not exceed Seven.

By G. A. Miller.

§ 1. Introduction.

Abstract group theory does not only owe its birth to the substitution groups of low degrees but many abstract group properties may be studied most advantageously by means of these substitution groups. The bond between these two kinds of groups of finite order is becoming more and more evident, and the difference between them is frequently only a matter of language. This is especially true as regards the transitive substitution groups and the abstract groups. In view of these facts it may be desirable to have a table by means of which one may readily obtain abstract definitions for all the substitution groups whose degrees do not exceed seven.

The fact that the same substitution group may frequently be defined abstractly in a large number of different ways constitutes an element of difficulty in the preparation of such a table. While we have tried to give one of the simplest definitions in each case, it must be admitted that in such matters simplicity is largely dependent upon the point of view. The definition which involves the smallest number of symbols does not always exhibit the properties of a group most clearly. Notwithstanding this obstacle it will doubtless often be helpful to have on hand at least one abstract definition of each one of the substitution groups in question. To increase the usefulness of the table we have added some details under the heading of Explanations. These are given just after the table and are especially extensive as regards the groups of the first few degrees.

As the cyclic group of order n may be represented by the n nth roots of unity, we shall define this group by the equation $s^n = 1$. The dihedral group of order $2m$ may be conveniently defined by $s_1^2 = 1$, $s_2^2 = 1$, $(s_1 s_2)^m = 1$; since it is generated by any two operators of order 2 whose product is of order m. This group is a special case of the one which may be obtained by extending any

abelian group (H) by means of an operator of order 2 which transforms every operator of H into its inverse. The order of the group (K) obtained in this way is evidently twice the order of H, and K involves only operators of order 2 in addition to those found in H. We proceed to give an abstract definition of K.

Consider the group generated by n operators of order 2 which are such that the product of any three of them is also of order 2. That is, we are considering $(s_1, s_2, \ldots, s_n)$ when these generators satisfy the conditions:

$$s_1^2 = s_2^2 = s_3^2 = \ldots = s_n^2 = (s_\alpha s_\beta s_\gamma)^2 = 1; \quad \alpha, \beta, \gamma = 1, 2, \ldots, n.$$

It is easy to see that each of these operators transforms into its inverse the product of any two of them. This fact follows directly from the equations:

$$s_\gamma s_\alpha s_\beta s_\gamma = s_\gamma \cdot s_\gamma s_\beta s_\alpha = s_\beta s_\alpha = (s_\alpha s_\beta)^{-1}.$$

Since each of the given n operators transforms the product of any two of them into its inverse, it results that the product of any two must be commutative with the product of any other two. That is, the operators which are the products of an even number of the operators $s_1, s_2, \ldots, s_n$ are commutative with each other, and hence they generate an abelian group H.

The operators $s_1, s_2, \ldots, s_n$ can clearly be so selected that the $n-1$ cyclic groups generated by $s_1s_2, s_1s_3, s_1s_4, \ldots, s_1s_n$ respectively are independent, and hence H may have $n-1$ invariants. Since any product involving an even number of factors from $s_1, s_2, \ldots, s_n$ may be formed by means of the given $n-1$ products, it results that H can not involve more than $n-1$ invariants when these invariants are so selected that each of them is divisible by all those which follow it. Hence we have proved the theorem:

If n operators of order 2 are such that the product of any three of them is also of order 2, they generate a group which involves an abelian subgroup composed of half the operators of the group, and each of the remaining operators is of order 2 and transforms each operator of this abelian subgroup into its inverse. Moreover, if this abelian group involves no more than $n-1$ invariants, the given group of twice its order can always be generated by n such operators of order 2.

If H is any subgroup of index ρ under G, all the operators of G may be uniquely represented in either of the following two ways:

$$\begin{aligned} G &= H + Hs_2 + Hs_3 + \ldots + Hs_\rho, \\ &= H + s_2H + s_3H + \ldots + s_\rho H. \end{aligned}$$

The sets $Hs_\alpha, s_\alpha H$ ($\alpha = 2, 3, \ldots, \rho$) are known respectively as the right and the left co-sets of G as regards H, and the sets obtained by adding H to these

co-sets are known as the augmented right and left co-sets respectively. That is, the augmented co-sets always include the subgroup with respect to which the sets are formed. These definitions are useful for the purpose of formulating a theorem which is very fundamental in finding suitable abstract definitions of known groups. This theorem may be stated as follows: *Two necessary and sufficient conditions that the augmented right co-sets*

$$H + Hs_2 + Hs_3 + \ldots + Hs_\rho$$

constitute a group are that they include the product of any two of the operators $s_2, s_3, \ldots, s_\rho$ *and that they also include the inverse of each operator in these co-sets.**

The fact that the conditions expressed in this theorem are necessary follows directly from the definition of a group. The fact that they are also sufficient may be established as follows. When these conditions are satisfied, the given right co-sets must include the following left co-sets:

$$s_2^{-1}H,\ s_3^{-1}H,\ \ldots,\ s_\rho^{-1}H.$$

Since no two operators of these co-sets are identical, the total number of operators in the given right co-sets are identical with those in these left co-sets. The product of any two operators in these right co-sets may be represented by $Hs_\alpha Hs_\beta$. As $s_\alpha H$ is contained in H or in a right co-set and hence also in the given augmented right co-sets, it results that these co-sets constitute a group. As a very special case of the given theorem we may state the fact that if we obtain only products of order 2 by multiplying all the operators of a group by a given operator this group and these products of order 2 must constitute a group whose order is twice the order of the original group.

Suppose now that $t_1, t_2, \ldots, t_\lambda$ is a set of generators of H and that $H + Hs_2 + Hs_3 + \ldots + Hs_\rho$ includes the inverses of each of the operators

$$t_{\alpha'} s_{\beta'}^{-1}; \qquad \alpha' = 1, 2, \ldots, \lambda;\ \beta' = 2, 3, \ldots, \rho.$$

It is also assumed that $s_2, s_3, \ldots, s_\rho$ satisfy the condition imposed on them in the preceding theorem. Any operator in the product $Hs_\alpha Hs_\beta$ may be written in the form

$$Hs_\alpha t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_\gamma^{\alpha_\gamma} \ldots s_\beta .$$

By successive reduction this assumes the form Hs_γ, and hence the given augmented right co-sets constitute a group also in this case. To illustrate this

*In the definition of co-sets it is implied that s_β is not contained in any of the sets $H, Hs_2, \ldots, Hs_\alpha$ $(\alpha < \beta)$; it is however not assumed that the augmented co-sets as regards a group (H) necessarily constitute a group.

corollary we shall employ it for the purpose of finding a simple abstract definition of the symmetric group of degree n as follows:

$$s_1^2 = s_2^2 = \ldots\ldots = s_{n-1}^2 = (s_j s_k)^3 = (s_i s_j s_i s_k)^2 = 1; \quad i \neq j \neq k;$$
$$i, j, k = 1, 2, \ldots\ldots, n-1.$$

That these equations define a group whose order is at least equal to that of the symmetric group of degree n results directly from the fact that they are satisfied when $s_1, s_2, \ldots\ldots, s_{n-1}$ are replaced respectively by the substitutions a_1a_n, a_2a_n, a_3a_n, $\ldots\ldots$, $a_{n-1}a_n$.

Let $t_1, t_2, \ldots\ldots, t_{n-2}$ be the transforms as regards s_{n-1} of $s_1, s_2, \ldots\ldots, s_{n-2}$, in order. It is evident that the t's satisfy the conditions imposed on the s's, if the latter satisfy these conditions, except that $i, j, k = 1, 2, \ldots\ldots, n-2$. Hence we may assume that these t's generate the symmetric group of degree $n-1$, and if we can prove that this assumption implies that the s's generate the symmetric group of degree n our theorem is clearly established. It is therefore only necessary to prove that if $H = (t_1, t_2, \ldots\ldots, t_{n-2})$ is the symmetric group of degree $n-1$ then must

$$H + Hs_1 + Hs_2 + \ldots\ldots + Hs_{n-1}$$

be the symmetric group of degree n. That these operators constitute a group whose order is n times that of H results directly from the following equations together with the corollary noted above:

$$s_\alpha s_\beta = t_\alpha t_\beta t_\alpha . s_\alpha \ (\alpha, \beta < n-1), \ s_{n-1}s_\alpha = t_\alpha s_{n-1}, \ s_\alpha s_{n-1} = t_\alpha s_\alpha;$$
$$s_\beta t_\alpha = s_\beta s_{n-1} s_\alpha s_{n-1} = t_\alpha s_\beta \ (\beta < n-1, \ \alpha \neq \beta); \ s_\beta t_\alpha = s_\alpha s_{n-1}, \ \beta = n-1;$$
$$s_\beta t_\alpha = s_{n-1}s_\alpha, \ \alpha = \beta.$$

As the s's generate a group whose order is n times that of H, whenever these s's satisfy the given conditions it results by induction that they generate the symmetric group of degree n. It should be observed that a somewhat different general abstract definition for the symmetric group of degree n as well as a general definition of the alternating group has been given by Moore* and others, and that the present definition has been given mainly for the purpose of illustrating the theorem in question. From the following list it will become apparent that it is frequently possible to give much briefer definitions in special cases.

In a similar manner we can obtain an abstract definition of the alternating

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group of degree n as a special case of the given theorem. To do this we may start with the following $n-1$ operators:

$$s_1^3 = s_2^2 = s_3^2 = \ldots\ldots = s_{n-2}^2 = (s_i s_j)^3 = (s_i s_j s_i s_k)^2 = 1\,; \quad i \neq j \neq k\,;$$
$$i, j, k = 1, 2, \ldots\ldots, n-2.$$

That these $n-2$ operators generate a group whose order is at least as large as that of the alternating group of degree n becomes evident if we observe that $s_1, s_2, s_3, \ldots\ldots, s_{n-2}$ may be replaced by the following substitutions, in order, in the given defining relations: $a_1a_2a_n,\ a_1a_2 \,.\, a_3a_n,\ a_1a_2 \,.\, a_4a_n,\ \ldots\ldots,\ a_1a_2 \,.\, a_{n-1}a_n$.

If we transform $s_1, s_1^{-1}, s_2, s_3, \ldots\ldots, s_{n-2}$ by the last one of these operators and represent the resulting operators, in order, by $t_1, t_1^{-1}, t_2, t_3, \ldots\ldots, t_{n-3}$, it is evident that these t's satisfy the conditions imposed on the s's. Hence we have only to prove that the latter generate the alternating group of degree n if the former generate this group of degree $n-1$. That the latter have this property follows directly from the equations:

$$s_\alpha s_\beta = t_\alpha t_\beta t_\alpha s_\alpha\ (1 < \alpha,\ \beta < n-2), \quad s_1 s_\beta = t_1^2 t_\beta t_1^2 s_1^2, \quad s_1 s_{n-2} = t_1^2 t_{n-2} t_1^2 s_{n-2},$$
$$s_1^2 s_\beta = t_1 t_\beta t_1 s_1, \quad s_1^2 s_{n-2} = t_1 s_1, \quad s_\beta s_1 = t_1 t_\beta t_1 s_\beta, \quad s_{n-2} s_1 = t_1 s_{n-2},$$
$$s_\beta s_1^2 = t_1^2 t_\beta t_1^2 s_\beta, \quad s_{n-2} s_1^2 = t_1^2 s_{n-2}, \quad s_{n-2} s_\beta = t_\beta s_{n-2}, \quad s_\beta s_{n-2} = t_\beta s_\beta\,;$$
$$s_\beta t_\alpha = t_\alpha s_\beta\ (1 < \alpha,\ \beta < n-2\,;\ \alpha \neq \beta), \quad s_{n-2} t_\alpha = s_\alpha s_{n-2},$$
$$s_\alpha t_\alpha = s_{n-2} s_\alpha, \quad s_1 t_\alpha = t_\alpha s_1^2, \quad s_1^2 t_\alpha = t_\alpha s_1, \quad s_1 t_1 = t_1^2 s_{n-2},$$
$$s_1^2 t_1 = t_1^2 s_1, \quad s_1 t_1^2 = t_1 s_1^2, \quad s_1^2 t_1^2 = t_1 s_{n-2}, \quad s_\alpha t_1 = t_1^2 s_\alpha,$$
$$s_\alpha t_1^2 = t_1 s_\alpha, \quad s_{n-2} t_1 = t_1^2 s_1^2, \quad s_{n-2} t_1^2 = t_1 s_1.$$

The similarity between the present abstract definition of the alternating group and the given abstract definition of the symmetric group should be noted.

The number of the different abstract groups which can be represented as substitution groups on seven or a smaller number of letters is 54.* Some of these groups can be represented in more than one way as substitution groups on seven or a smaller number of letters. In what follows we shall use only one such representation; that is, we shall consider only one of a set of simply isomorphic groups, and we shall represent this on the smallest possible number of letters. If this is done, the numbers of different abstract groups which may be represented as substitution groups of the various degrees from 2 to 7 are respectively 1, 2, 5, 7, 13, 26. The notation employed is practically the same as was used in the list published in the AMERICAN JOURNAL OF MATHEMATICS to which we have referred. The only change is the omission of the abbreviation

*AMERICAN JOURNAL OF MATHEMATICS, Vol. XXI (1899), p. 326.

cyc after the parenthesis enclosing the generator of a cyclic group. This omission seems to be a step towards simplicity and uniformity in substitution notation.

§ 2. LIST OF GROUPS.

Degree Two.

Order.	No.	Notation.	Abstract Definition.
2	1	(ab)	$s_1^2 = 1$

Degree Three.

3	1	(abc)	$s_1^3 = 1$
6	1	(abc) all	$s_1^2 = s_2^2 = (s_1s_2)^3 = 1$

Degree Four.

4	1	$(ab)(cd)$	$s_1^2 = s_2^2 = (s_1s_2)^2 = 1$
	2	$(abcd)$	$s_1^4 = 1$
8	1	$(abcd)_8$	$s_1^2 = s_2^2 = (s_1s_2)^4 = 1$
12	1	$(abcd)$ pos	$s_1^2 = s_2^3 = (s_1s_2)^3 = 1$
24	1	$(abcd)$ all	$s_1^2 = s_2^3 = (s_1s_2)^4 = 1$

Degree Five.

5	1	$(abcde)$	$s_1^5 = 1$
6	1	$(abc)(de)$	$s_1^6 = 1$
10	1	$(abcde)_{10}$	$s_1^2 = s_2^2 = (s_1s_2)^5 = 1$
12	1	(abc) all (de)	$s_1^2 = s_2^2 = (s_1s_2)^6 = 1$
20	1	$(abcde)_{20}$	$s_1^4 = s_2^5 = s_1^3s_2s_1s_2^3 = 1$
60	1	$(abcde)$ pos	$s_1^2 = s_2^3 = (s_1s_2)^5 = 1$
120	1	$(abcde)$ all	$s_1^4 = s_2^5 = (s_1^2s_2)^3 = (s_1s_2^3)^2 = 1$

Degree Six.

8	1	$(ab)(cd)(ef)$	$s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^2 = (s_1s_3)^2 = (s_2s_3)^2 = 1$
	2	$(abcd)(ef)$	$s_1^4 = s_2^2 = s_1^3s_2s_1s_2 = 1$
9	1	$(abc)(def)$	$s_1^3 = s_2^3 = s_1^2s_2^2s_1s_2 = 1$
16	1	$(abcd)_8(ef)$	$s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^2 = (s_1s_3)^2 = (s_2s_3)^4 = 1$
18	1	(abc) all (def)	$s_1^2 = s_2^2 = s_3^3 = (s_1s_2)^3 = s_1s_3s_1s_3^2 = s_2s_3s_2s_3^2 = 1$
	2	$\{(abc)$ all (def) all$\}$ pos	$s_1^2 = s_2^2 = s_3^2 = (s_1s_2s_3)^2 = (s_1s_2)^3 = (s_1s_3)^3 = 1$
24	1	$(abcd)$ pos (ef)	$s_1^2 = s_2^3 = (s_1s_2^2s_1s_2)^2 = 1$
36	1	(abc) all (def) all	$s_1^6 = s_2^6 = (s_1s_2)^2 = s_1^5s_2^2s_1s_2^2 = s_2^5s_1^2s_2s_1^2 = 1$
	2	$(abcdef)_{36}$	$s_1^4 = s_2^4 = (s_1s_2)^2 = (s_1^2s_2^2)^3 = (s_1s_2^3)^3 = 1$
48	1	$(abcd)$ all (ef)	$s_1^2 = s_2^6 = (s_1s_2^2)^4 = (s_1s_2^3)^2 = 1$
72	1	$(abcdef)_{72}$	$s_1^2 = s_2^4 = (s_1s_2)^6 = (s_1s_2^3s_1s_2)^2 = 1$

360	1	$(abcdef)$ pos	$s_1^3 = s_2^4 = (s_1 s_2^2)^4 = (s_1 s_2^3 s_1^2 s_2)^2 = 1$
720	1	$(abcdef)$ all	$s_1^6 = s_2^4 = (s_1^2 s_2^2)^4 = (s_1^2 s_2^3 s_1^4 s_2)^2$ $= s_1^5 s_2^3 s_1 (s_1^2 s_2)^2 s_2 (s_1^2 s_2)^3 s_2 = 1$

Degree Seven.

7	1	$(abcdefg)$	$s_1^7 = 1$
10	1	$(abcde)(fg)$	$s_1^{10} = 1$
12	1	$(ab)(cd)(efg)$	$s_1^2 = s_2^6 = s_1 s_2^5 s_1 s_2 = 1$
	2	$(abcd)(efg)$	$s_1^{12} = 1$
	3	$\{(abcd)(efg)$ all$\}$ pos	$s_1^4 = s_2^3 = s_1^3 s_2 s_1 s_2 = 1$
14	1	$(abcdefg)_{14}$	$s_1^2 = s_2^2 = (s_1 s_2)^7 = 1$
20	1	$(abcde)_{10}(fg)$	$s_1^2 = s_2^2 = (s_1 s_2)^{10} = 1$
21	1	$(abcdefg)_{21}$	$s_1^3 = s_2^7 = s_1^2 s_2 s_1 s_2^5 = 1$
24	1	$(ab)(cd)(efg)$ all	$s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^6 = (s_1 s_3)^2 = (s_2 s_3)^2 = 1$
	2	$(abcd)(efg)$ all	$s_1^2 = s_2^{12} = s_1 s_2 s_1 s_2^7 = 1$
	3	$(abcd)_8(efg)$	$s_1^2 = s_2^{12} = s_1 s_2 s_1 s_2^5 = 1$
	4	$\{(abcd)_8$ com (efg) all$\}$ dim	$s_1^4 = s_2^6 = s_1^2 s_2 s_1^2 s_2^5 = s_1^3 s_2 s_1 s_2 s_1^2 = 1$
	5	$\{(abcd)_8$ cyc (efg) all$\}$ dim	$s_1^2 = s_2^2 = (s_1 s_2)^{12} = 1$
36	1	$(abcd)$ pos (efg)	$s_1^6 = s_2^3 = (s_1^3 s_2)^3 = s_1^4 s_2 s_1^2 s_2^2 = 1$
40	1	$(abcde)_{20}(fg)$	$s_1^4 = s_2^{10} = s_1^3 s_2 s_1 s_2^3 = 1$
42	1	$(abcdefg)_{42}$	$s_1^6 = s_2^7 = s_1^5 s_2 s_1 s_2^4 = 1$
48	1	$(abcd)_8(efg)$ all	$s_1^2 = s_2^2 = s_3^{12} = (s_1 s_2)^2 = s_1 s_3 s_1 s_3^7 = s_2 s_3 s_2 s_3^5 = 1$
72	1	$(abcd)$ all (efg)	$s_1^3 = s_2^{12} = (s_1 s_2^3)^2 = s_2^8 s_1^2 s_2^4 s_1 = 1$
	2	$(abcd)$ pos (efg) all	$s_1^6 = s_2^6 = (s_1^2 s_2^3)^3 = (s_1^3 s_2^2)^2 = (s_1^3 s_2^3)^2 = s_1^2 s_2^2 s_1^4 s_2^4 = 1$
	3	$\{(abcd)$ all (efg) all$\}$ pos	$s_1^4 = s_2^2 = s_3^3 = (s_1 s_2)^3 = (s_2 s_3)^2 = s_1^3 s_3 s_1 s_3 = 1$
120	1	$(abcde)$ pos (ef)	$s_1^2 = s_2^6 = (s_1 s_2^2)^5 = (s_1 s_2^3)^2 = 1$
144	1	$(abcd)$ all (efg) all	$s_1^6 = s_2^{12} = (s_1^2 s_2^3)^2 = (s_1^3 s_2^4)^2 = s_1^2 s_2^4 s_1^4 s_2^8 = s_1^3 s_2^3 s_1^3 s_2^9 = 1$
168	1	$(abcdefg)_{168}$	$s_1^7 = s_2^3 = (s_1 s_2)^2 = (s_2 s_1^5)^4 = 1$
240	1	$(abcde)$ all (fg)	$s_1^4 = s_2^{10} = (s_1^2 s_2^2)^3 = (s_1 s_2^6)^2 = s_1^3 s_2^5 s_1 s_2^5 = 1$
2520	1	$(abcdefg)$ pos	$s_1^3 = s_2^2 = s_3^2 = s_4^2 = s_5^2 = (s_i s_j)^3 = (s_i s_j s_i s_k)^2 = 1;$ $i \neq j \neq k;\ i, j, k = 1, 2, \ldots, 5$
5040	1	$(abcdefg)$ all	$s_i^2 = (s_j s_k)^3 = (s_i s_j s_i s_k)^2 = 1,\ i \neq j \neq k,$ $i, j, k = 1, 2, \ldots, 6$

§ 3. Explanations.

Groups of Degrees Three and Four.

The two groups of degree 3 are respectively cyclic and dihedral, while those of degree 4 are cyclic, dihedral, or of genus zero. Hence all of these groups belong to well-known categories of abstract groups. The symmetric group of

degree 3 is the smallest non-abelian dihedral group. It may also be defined as the only non-abelian group involving only two complete sets of conjugates besides the identity, as the holomorph of the group of order 3, as the only non-abelian group that involves only three operators of order 2 and is generated by them, as the only non-abelian group that has the symmetric group of order 6 for its group of isomorphisms,* as the group of movements of the equilateral triangle, as the group generated by subtracting from a given number x_1 and by dividing x_1^2, or by multiplying by 2 and by adding unity, if we start with a given parameter and reduce the results modulo 3, as the only non-abelian group in which the number of complete sets of conjugate operators (excluding the identity) is exactly equal to the number of distinct prime factors in the order of the group.†

The non-cyclic group of order 4 is known under a number of different names. Among these are the following: the quadratic group, the anharmonic ratio group, the rectangle group, the axial group, the four-group. It is the non-cyclic group of lowest order, and it presents itself in various parts of mathematics as these different names imply. For instance, in the ordinary complex plane it is the group generated by the operations of inversion and reflection on the x-axis; on a line it is the group of the permutation of four points such that the anharmonic ratio of these points is left unchanged; in number theory it is the group formed by multiplying the numbers which are prime to 8 and reducing them and their products modulo 8; etc. the cyclic group of order 4 is perhaps best known as the one which is generated by $\sqrt{-1}$.

The substitution group of degree 4 and of order 8 is dihedral and is commonly called the *octic* group. It is the holomorph of the cyclic group of order 4 as well as the group of movements of a square. It is the only group of order 2^m which contains exactly five operators of order 2 and is generated by them. It is also generated by the two operations of starting with a given angle and taking the supplement and the complement as well as by the two operations of subtracting from x_1 and of dividing $x_1^2/2$. The alternating group of degree 4 is commonly known as the *tetrahedral* group, since it is the group of movements of the regular tetrahedron. It does not contain any subgroup of order 6 and is the group of lowest order that does not contain a subgroup whose order is an arbitrary divisor of the order of the group. The symmetric group of degree 4 is

* *Transactions of the American Mathematical Society*, Vol. I (1900), p. 399.

† *Ibid.*, Vol. V (1904), p. 505.

called the *octahedral* group, since it is the group of movements of the regular octahedron. The following are some of its characteristic properties: It is the holomorph of the axial group; it is the group of movements of the cube; it contains exactly nine operators of order 2 which generate it and are such that the order of the product of at least two of them is 4, while no two of them have a product whose order exceeds 4; it is the smallest group that has a non-abelian commutator subgroup and is generated by two operators of orders 2 and 3 respectively having a commutator of order 3;* it is generated by three cyclic subgroups of order 4 which are non-invariant and do not involve a common subgroup of order 2 nor generate other cyclic subgroups of order 4.†

Groups of Degrees Five and Six.

As all the substitution groups of degree 5, with the exception of the alternating and the symmetric group, are cyclic, dihedral or metacyclic, their properties may have been sufficiently explained in what precedes. Moreover, the icosahedral group is so well known that it seems unnecessary to enter into a discussion of its properties. The given abstract definition of the symmetric group may, however, deserve a word of explanation. It is clear that s_1^2, s_2 satisfy the ordinary conditions of generators of the icosahedral group and that this group is transformed into itself by s_1, since the last relation implies that $s_1^{-1}s_2^2s_1 = s_2^3s_1^2$. It therefore results that (s_1, s_2) is of order 120. Since s_1, s_2 may be replaced respectively by the substitutions *aceb* and *abcde*, it must be the symmetric group of this order.

The given abstract definition of the substitution group of degree 6 and order 24 is in accord with the theorem,‡ *If an operator of order* 2 *and an operator of order* 3 *have a commutator of order* 2, *they generate either the tetrahedral group or the direct product of this group and a group of order* 2. To verify the given abstract definition of the second group of order 36 we may proceed as follows: The two conjugate operators $s_1^2s_2^2$, $s_1s_2^2s_1$ are commutative, since their commutator is $(s_2^2s_1)^4 = 1$. Hence they generate a group of order 9. This is evidently invariant under (s_1, s_2) and each of its operators is transformed into its inverse by each of the two operators s_1^2, s_2^2. The group of order 18 generated by s_1^2, $s_1^2s_2^2$, $s_1s_2^2s_1$ is also invariant under (s_1, s_2), since $s_2^3s_1^2s_2 = s_1s_2^2s_1 \cdot s_2^2$. Hence it results that $(s_1, s_1^2s_2^2, s_1s_2^2s_1)$ is of order 36. It must include s_2, since $(s_1s_2^3)^3 = 1 = s_1s_2^2s_1^3s_2^2s_1s_2^3$. If we omit the condition $(s_1s_2^3)^3 = 1$, the remaining conditions define the direct product of this group of order 36 and the group of order 2.

* *Transactions of the American Mathematical Society*, Vol. IX (1908), p. 76.

† *Mathematische Annalen*, Vol. LXIV (1907), p. 344.

‡ *Transactions of the American Mathematical Society*, Vol. IX (1908), p. 68.

The given abstract definition of the group of order 48 results immediately from the fact that it is the direct product of the octahedral group and the group of order 2. The group of order 72 is evidently generated by the substitutions *ab* and *afcd* . *be*, and these two substitutions satisfy the given abstract definitions. It is therefore only necessary to prove that if s_1, s_2 satisfy these defining relations they can not generate a group whose order exceeds 72. Let $t_1 = (s_1s_2)^2$, and observe that t_1 and $s_1t_1s_1$ are commutative. The group of order 9 $(t_1, s_1t_1s_1)$ is evidently invariant under (s_1, s_2). It is also easy to verify that the subgroup of order 18 $(t_1, s_1t_1s_1, s_1s_2^3s_1s_2)$ is invariant, and that this is identical with $(t_1, s_1t_1s_1, s_2^2)$. As a final step we may observe that $(t_1, s_1t_1s_1, s_2^2, s_1)$ is an invariant subgroup of order 36, since $s_2^3s_1s_2 = s_2^3s_1s_2s_1 \, . \, s_1$. The given abstract definition for the alternating group of degree 6 is the one discussed by Dickson in the *Bulletin of the American Mathematical Society*, Vol. IX (1903), p. 303. The abstract definition of the symmetric group of degree 6 can be directly obtained from this definition of the alternating group.

Groups of Degree Seven.

The given abstract definition of the group of order 36 results from the ordinary definition of the tetrahedral group and the fact that this group of order 36 is the direct product of the tetrahedral group and the group of order 3. The definition of the group of order 48 may be illustrated by means of the substitutions $s_1 = de$, $s_2 = ac$, $s_3 = abcd \, . \, efg$. The definition of the first group of order 72 results directly from that of the octahedral group, while that of the second becomes evident if we observe that (s_1^2, s_2^3) is the tetrahedral group and that (s_1^3, s_2^2) is the symmetric group of order 6. In the definition of the third group of order 72 it may be observed that (s_1, s_3) is the octahedral group, in accord with the common definition of this group. The definition of the group of order 120 results immediately from that of the icosahedral group.

The given abstract definition of the group of order 144 may be verified by observing that (s_1^2, s_2^3) is the octahedral group, while (s_1^3, s_2^4) is the symmetric group of order 6, and each operator of one of these groups is commutative with every operator of the other. The defining relations of the simple group of order 168 agree with those given by Dyck in the *Mathematische Annalen*, Vol. XX (1882), p. 41. The definition of the group of order 240 results directly from that of the symmetric group of degree 5, while the definitions of the alternating and the symmetric group are illustrations of the general definitions of these groups to which we referred in the Introduction.

UNIVERSITY OF ILLINOIS.